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New Oscillation Criteria for Second-Order Neutral Delay Dynamic Equations

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Abstract

We establish some new oscillation criteria for a class of second-order nonlinear neutral delay dynamic equations using a couple of Riccati substitutions. Our main results not only complement those related results in the literature, but also improve some known results for second-order delay dynamic equations without neutral terms.

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1 Introduction

In this paper, we introduce new sufficient conditions for the oscillation of solutions of the neutral dynamic equation

$$\left[r(t)\varphi_\alpha(z^\Delta(t))\right]^\Delta + q(t)f(\varphi_\beta x(\delta(t))) = 0 \quad \text{for } t \in [t_0,\infty)_\mathbb{T} \tag{1}$$

where $z(t) := x(t) + p(t)x(\tau(t))$, $\varphi_\gamma(\lambda) := \operatorname{sgn}(\lambda)|\lambda|^\gamma$ for $\lambda \in \mathbb{R}$ and $\gamma \in \mathbb{R}^+$, $\alpha, \beta \in \mathbb{R}^+$. We assume the following conditions .

(H$_1$) $r \in \mathrm{C}^1_{rd}([t_0,\infty)_\mathbb{T},\mathbb{R})$, $\int_{t_0}^\infty r^{-1/\alpha}\Delta s = \infty$.

(H$_2$) $\tau, \delta \in \mathrm{C}^1_{rd}([t_0,\infty)_\mathbb{T},\mathbb{T})$, $\tau(t) \le t$, $\delta(t) \le t$, $\tau \circ \delta = \delta \circ \tau$, $\lim_{t\to\infty}\tau(t) = \infty$, $\lim_{t\to\infty}\delta(t) = \infty$, $\tau^\Delta(t) \ge \tau_0$, and $\delta^\Delta(t) > 0$ where τ_0 is a constant.

(H$_3$) $p, q \in \mathrm{C}_{rd}([t_0,\infty)_\mathbb{T},\mathbb{R})$, $0 \le p(t) \le p_0 < \infty$, $q(t) \ge 0$ and $q(t)$ is not identically zero for large t.

(H$_4$) $f \in C(\mathbb{T},\mathbb{T})$, and there exists a positive constant k such that $\frac{f(x)}{x^\alpha} \ge K$ for all $x \ne 0$.

The theory of time scales was introduced by Hilger (see [5]) in 1988 in order to unify continuous and discrete analysis. A *time scale*, which inherits the standard topology on $\mathbb{R}$, is a nonempty closed subset of reals. Here, and later throughout this paper, a time scale will be denoted by the symbol $\mathbb{T}$, and the intervals with a subscript $\mathbb{T}$ are used to denote the intersection of the usual interval with $\mathbb{T}$. For $t \in \mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf(t,\infty)_\mathbb{T}$, while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup(-\infty,t)_\mathbb{T}$, and the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined to be $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is called *right-dense* if $\sigma(t) = t$ and/or equivalently $\mu(t) = 0$ holds; otherwise, it is called *right-scattered*, and similarly *left-dense* and *left-scattered points* are defined with respect to the backward jump operator. The set of all such *rd*-continuous functions is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ which are differentiable and whose derivative is an *rd*-continuous function is denoted by $C^1_{rd}(\mathbb{T},\mathbb{R})$. For some concepts related to the notion of time scales, see [4].

By a solution of (1) we mean a nontrivial function $x \in \mathrm{C}_{rd}([T_x,\infty)_\mathbb{T},\mathbb{R})$, where $T_x \in [t_0,\infty)_\mathbb{T}$, which has the property that $\left[r(t)\varphi_\alpha((x+p\cdot x\circ\tau)^\Delta(t))\right] \in \mathrm{C}^1_{rd}([T_x,\infty)_\mathbb{T},\mathbb{R})$ and satisfies (1) identically on $[T_x,\infty)_\mathbb{T}$. A solution x of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (1) is called oscillatory if all its solutions oscillate.

In recent years there has been much research activities concerning the oscillation of solutions of several classes of neutral dynamic equations, see [7,9,10,12] Several papers are devoted to study the cases in which $0 < p(t) < 1$ and $0 < p(t) \leq p_0 < \infty$, for instance, in case of $\mathbb{T} = \mathbb{R}$ Baculíková and Džurina [2] studied the second order neutral differential equation

$$\left[r(t)(x(t) + p(t)x(\tau(t)))'\right]' + q(t)f(x(\delta(t))) = 0.$$

They presented new oscillation criteria, where they replaced the traditional restriction $0 \leq p(t) < 1$ by $0 < p(t) \leq p_0 < \infty$ and $\delta(t) \leq \tau(t) < t$. They use new comparison theorems, that enable them to reduce the problem of the oscillation of the second order equation to the oscillation of the first order equation.

In [13] Zhang *et al.* introduce new oscillation criteria for the class of second order dynamic equations of the type,

$$\left[r(t)(x(t) + p(t)x(\tau(t)))^{\Delta}\right]^{\Delta} + q(t)f(x(\delta(t))) = 0.$$

Under the conditions $0 < p(t) \leq p_0 < \infty$, $\int_{t_0}^{\infty} \frac{\Delta s}{r(s)} = \infty$ and $\int_{t_0}^{\infty} \frac{\Delta s}{r(s)} < \infty$. In [3] Baculíková and Džurina studied the oscillation of the second-order neutral differential equations of the form

$$\left[r(t)\big((x(t) + p(t)x(\tau(t)))^{\alpha}\big)\right]^{\Delta} + q(t)x^{\beta}(\delta(t)) = 0.$$

where α, β are the ratios of two positive odd integers, $0 \leq p(t) < \infty$, $\tau(t) \geq t$, $\delta(t) \leq \tau(t) \leq t$, but they did not consider the case $\tau(t) \leq \delta(t) \leq t$.

Our aim in this paper is to obtain some new sufficient conditions for (1) and improve the results of [1, 3, 13, 14] .

Now, we present some known results, which needed in the proof of our main results.

Theorem 1.1. [4] *Assume that* $v : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\tilde{\mathbb{T}} := v(\mathbb{T})$ *is a time scale. Let* $y : \tilde{\mathbb{T}} \to \mathbb{R}$. *If* $y^{\tilde{\Delta}}[v(t)]$ *and* $v^{\Delta}(t)$ *exist for* $t \in \mathbb{T}_k$, *then*

$$(y[v(t)])^{\Delta} = y^{\tilde{\Delta}}[v(t)]v^{\Delta}(t).$$

Lemma 1.2. [11] *If* $X \geq 0$, $Y \geq 0$, *and* $0 < \lambda \leq 1$, *then*
$X^{\lambda} + Y^{\lambda} \geq (X + Y)^{\lambda}$.

Lemma 1.3. [3] *If* $X \geq 0$, $Y \geq 0$, *and* $\lambda \geq 1$, *then*
$X^{\lambda} + Y^{\lambda} \geq 2^{1-\lambda}(X + Y)^{\lambda}$.

Lemma 1.4. [6] *If* $B > 0$, $A > 0$, *and* $\alpha > 0$, *then*

$$Au - Bu^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{A^{\alpha+1}}{B^{\alpha}}.$$

2 Main results

For convenience, we define

$$K_0 = \begin{cases} K & 0 < \beta \leq 1, \\ 2^{1-\beta}K, & \beta > 1. \end{cases}, \quad \rho_+^{\Delta}(t) = \max\{0, \rho^{\Delta}(t)\}.$$

The following theorem introduces a new oscillation criterion when $\delta(t) \geq \tau(t)$

Theorem 2.1. *Assume that* (H$_1$)*-*(H$_4$) *and* $\delta(t) \geq \tau(t)$ *are satisfied. If there exists a function* $\rho \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ *such that for all constants* $\lambda_1, \lambda_2 > 0$*,we have*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(K_0\rho(\zeta)Q(\zeta) - \left(1 + \frac{p_0^{\beta}}{\tau_0}\right) \frac{(\alpha/\beta)^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{(\rho_+^{\Delta}(\zeta))^{\alpha+1} r(\tau(\zeta))}{\tau_0^{\alpha}\lambda_1^{\beta-\alpha}\rho^{\frac{1}{\alpha}}(\zeta)} \right) \Delta\zeta \quad \alpha \leq \beta \tag{2}$$

or

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(K_0\rho(\zeta)Q(\zeta) - \left(1 + \frac{p_0^{\beta}}{\tau_0}\right) \frac{(\rho_+^{\Delta}(\zeta))^{\frac{\beta+1}{\beta}} \lambda_2^{\frac{\alpha-\beta}{\alpha}} r^{\beta/\alpha}(\tau(\zeta))}{(\beta+1)^{\beta+1}\tau_0^{\beta}\rho^{\beta}(\zeta)} \right) \Delta\zeta \quad \alpha > \beta, \tag{3}$$

where $Q(t) = \min\{q(t), q(\tau(t))\}$*, then Eq.* (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1) with $x(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$, then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\delta(t)) > 0$, for all $t \in [t_0, \infty)_{\mathbb{T}}$. By the definition of $z(t)$, we have $z > 0$ and $z(t) \geq x(t)$, $t \geq t_1$. From (1), we have

$$[r(t)\varphi_{\alpha}(z^{\Delta}(t))]^{\Delta} \leq -Kq(t)x^{\beta}(\delta(t)) \leq 0. \tag{4}$$

From (H$_1$) and (4), one can easily obtain $z^{\Delta}(t) > 0$. then (4) becomes

$$[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta} \leq -Kq(t)x^{\beta}(\delta(t)) \leq 0. \tag{5}$$

It follows from Theorem 1.1 that $\left[r(\tau(t))(z^{\Delta}(\tau(t)))^{\alpha}\right]^{\Delta} = \left[r(t)(z^{\Delta}(t))^{\alpha}\right]^{\Delta}\tau^{\Delta}(t)$, that there exists a $t_2 \geq T$ such that

$$\frac{\left[r(\tau(t))(z^{\Delta}(\tau(t)))^{\alpha}\right]^{\Delta}}{\tau^{\Delta}(t)} \leq -Kq(\tau(t))x^{\beta}(\delta(\tau(t))).$$

But since $\tau^{\Delta}(t) \geq \tau_0 > 0$, we get for, $t \geq t_2$,

$$\frac{1}{\tau_0}\left[r(\tau(t))(z^{\Delta n-1}(\tau(t)))^{\alpha}\right]^{\Delta} \leq -Kq(\tau(t))x^{\alpha}(\delta(\tau(t))). \tag{6}$$

Combining (5) and (6), we obtain

$$\begin{aligned}[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}+&\frac{p_0^{\beta}}{\tau_0}\left[r(\tau(t))(z^{\Delta n-1}(\tau(t)))^{\alpha}\right]^{\Delta}\\ &+Kq(t)x^{\beta}(\delta(t))+p_0^{\beta}Kq(\tau(t))x^{\beta}(\delta(\tau(t)))\leq 0 \qquad (7)\end{aligned}$$

Assume that $0<\beta\leq 1$. Since $\delta\circ\tau=\tau\circ\delta$ and Lemma 1.2, we get

$$\begin{aligned}[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}+\frac{p_0^{\beta}}{\tau_0}\left[r(\tau(t))(z^{\Delta}(\tau(t)))^{\beta}\right]^{\Delta}\leq&-Kq(t)x^{\beta}(\delta(t))-p_0^{\beta}Kq(\tau(t))x^{\alpha}(\delta(\tau(t)))\\ \leq&-KQ(t)[x^{\beta}(\delta(t))+x^{\beta}(\delta(\tau(t)))]\\ \leq&-KQ(t)[x(\delta(t))+x(\delta(\tau(t)))]^{\beta}\\ \leq&-KQ(t)z^{\beta}(\delta(t)). \qquad (8)\end{aligned}$$

Now, if $\beta>1$. Similarly, in view of Lemma 1.3, we have

$$\begin{aligned}[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}+\frac{p_0^{\beta}}{\tau_0}\left[r(\tau(t))(z^{\Delta}(\tau(t)))^{\beta}\right]^{\Delta}\leq&-KQ(t)[x^{\beta}(\delta(t))+x^{\beta}(\delta(\tau(t)))]\\ \leq&-2^{1-\beta}KQ(t)[x(\delta(t))+x(\delta(\tau(t)))]^{\beta}\\ \leq&-2^{1-\beta}KQ(t)z^{\beta}(\delta(t)). \qquad (9)\end{aligned}$$

It follows from (8) and (9) that

$$[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}+\frac{p_0^{\beta}}{\tau_0}\left[r(\tau(t))(z^{\Delta}(\tau(t)))^{\beta}\right]^{\Delta}\leq -K_0Q(t)z^{\beta}(\delta(t)) \qquad (10)$$

Now, we define a Riccati substitution

$$\omega(t):=\rho(t)\frac{r(t)(z^{\Delta}(t))^{\alpha}}{z^{\beta}(\tau(t))}\quad \text{for all} t\in[t_1,\infty)_{\mathbb{T}} \qquad (11)$$

It is clear that $\omega>0$ for all $t\geq t_1$, and

$$\begin{aligned}\omega^{\Delta}(t)=&\frac{\rho(t)}{z^{\beta}(\tau(t))}[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}+[r(t)(z^{\Delta}(t))^{\alpha}]^{\sigma}\left[\frac{\rho(t)}{z^{\beta}(\tau(t))}\right]^{\Delta}\\ =&\rho(t)\frac{[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}}{z^{\beta}(\tau(t))}+\frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t))-\beta\tau^{\Delta}(t)\frac{\rho(t)}{\rho(\sigma(t))}\frac{z^{\Delta}(\tau(t))}{z(\tau(t))}\omega(\sigma(t)). \quad (12)\end{aligned}$$

Since $r(t)(z^{\Delta}(t))^{\alpha}$ decreasing, then $r(t)(z^{\Delta}(t))^{\alpha}\leq r(\tau(t))(z^{\Delta}(\tau(t)))^{\alpha}$, i.e.,

$$z^{\Delta}(\tau(t))\geq\left(\frac{r(t)}{r(\tau(t))}\right)^{1/\alpha}z^{\Delta}(t). \qquad (13)$$

This with (12) leads to

$$\omega^{\Delta}(t)\leq\rho(t)\frac{[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}}{z^{\beta}(\tau(t))}+\frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t))-\frac{\beta\tau_0\rho(t)}{\rho(\sigma(t))}\left(\frac{r(t)}{r(\tau(t))}\right)^{1/\alpha}\frac{z^{\Delta}(t)}{z(\tau(t))}\omega(\sigma(t)) \qquad (14)$$

Since $\beta \geq \alpha$, and $z^{\Delta}(t) > 0$, then there exists a constant $\lambda_1 > 0$ such that $z(t) \geq z(\tau(t)) \geq \lambda_1$. Using (13) and (14), we get

$$\omega^{\Delta}(t) \leq \rho(t)\frac{[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}}{z^{\beta}(\tau(t))}+\frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t))-\beta\tau_0\frac{\rho(t)\lambda_1^{\frac{\beta-\alpha}{\alpha}}}{\rho^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\tau(t))}\omega^{(\alpha+1)/\alpha}(\sigma(t)) \tag{15}$$

Applying Lemma 1.4 , we obtain

$$\omega^{\Delta}(t) \leq \rho(t)\frac{[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}}{z^{\beta}(\tau(t))} + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(\rho^{\Delta}(t))^{\alpha+1}r(\tau(t))}{\beta^{\alpha}\tau_0^{\alpha}\lambda_1^{\beta-\alpha}\rho^{\frac{1}{\alpha}}(t)} \tag{16}$$

Similarly , define another Riccati substitution

$$\nu(t) := \rho(t)\frac{r(\tau(t))(z^{\Delta}(\tau(t)))^{\alpha}}{z^{\beta}(\tau(t))} \quad \text{for all} t \in [t_1, \infty)_{\mathbb{T}}. \tag{17}$$

Then we have $\nu(t) > 0$. Differentiating (17), by $[r(\tau(t))(z^{\Delta}(\tau(t)))^{\alpha}] \geq [r(\tau(t))(z^{\Delta}(\tau(t)))^{\alpha}]^{\sigma} > 0$ and $z(\tau^{\sigma}(t)) \geq \lambda_1$

$$\begin{aligned}\nu^{\Delta}(t) =&\frac{\rho(t)}{z^{\beta}(\tau(t))}[r(\tau(t))(z^{\Delta}(\tau(t)))^{\alpha}]^{\Delta} + [r(\tau(t))(z^{\Delta}(\tau(t)))^{\alpha}]^{\sigma}\left[\frac{\rho(t)}{z^{\beta}(\tau(t))}\right]^{\Delta} \\ =&\rho(t)\frac{[r(\tau(t))(z^{\Delta}(\tau(t)))^{\alpha}]^{\Delta}}{z^{\beta}(\tau(t))} + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\nu(\sigma(t)) - \frac{\beta\tau_0\rho(t)\lambda_1^{\frac{\beta-\alpha}{\alpha}}}{\rho^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\tau(t))}\nu^{\frac{\alpha+1}{\alpha}}(\sigma(t))\end{aligned}$$

Applying Lemma 1.4 , we get

$$\nu^{\Delta}(t) \leq \rho(t)\frac{[r(\tau(t))(z^{\Delta}(\tau(t)))^{\alpha}]^{\Delta}}{z^{\beta}(\tau(t))} + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(\rho^{\Delta}(t))^{\alpha+1}r(\tau(t))}{\beta^{\alpha}\tau_0^{\alpha}\lambda_1^{\beta-\alpha}\rho^{\frac{1}{\alpha}}(t)} \tag{18}$$

Combining (16) and (18), we conclude that

$$\begin{aligned}\omega^{\Delta}(t) + \frac{p_0^{\beta}}{\tau_0}\nu^{\Delta}(t) \leq& \frac{\rho(t)}{z^{\beta}(\tau(t))}\left([r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta} + \frac{p_0^{\beta}}{\tau_0}[r(\tau(t))(z^{\Delta}(\tau(t)))^{\alpha}]^{\Delta}\right) \\ &+ \left(1+\frac{p_0^{\beta}}{\tau_0}\right)\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(\rho^{\Delta}(t))^{\alpha+1}r(\tau(t))}{\beta^{\alpha}\tau_0^{\alpha}\lambda_1^{\beta-\alpha}\rho^{\frac{1}{\alpha}}(t)}\end{aligned} \tag{19}$$

Recalling (10), implies

$$\omega^{\Delta}(t) + \frac{p_0^{\beta}}{\tau_0}\nu^{\Delta}(t) \leq -K_0\rho(t)Q(t)\frac{z^{\beta}(\delta(t))}{z^{\beta}(\tau(t))} + \left(1+\frac{p_0^{\beta}}{\tau_0}\right)\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(\rho^{\Delta}(t))^{\alpha+1}r(\tau(t))}{\beta^{\alpha}\tau_0^{\alpha}\lambda_1^{\beta-\alpha}\rho^{\frac{1}{\alpha}}(t)}. \tag{20}$$

Since $\delta(t) \geq \tau(t)$ and $z^{\Delta}(t) > 0$, then $z(\delta(t)) \geq z(\tau(t))$. This leads to

$$\omega^{\Delta}(t) + \frac{p_0^{\beta}}{\tau_0}\nu^{\Delta}(t) \leq -K_0\rho(t)Q(t) + \left(1+\frac{p_0^{\beta}}{\tau_0}\right)\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(\rho^{\Delta}(t))^{\alpha+1}r(\tau(t))}{\beta^{\alpha}\tau_0^{\alpha}\lambda_1^{\beta-\alpha}\rho^{\frac{1}{\alpha}}(t)}. \tag{21}$$

Integrating (21) from t_1 to t, we see that

$$\int_{t_1}^{t}\left(K_0\rho(\zeta)Q(\zeta)-\left(1+\frac{p_0^{\beta}}{\tau_0}\right)\frac{(\alpha/\beta)^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(\rho^{\Delta}(\zeta))^{\alpha+1}r(\tau(\zeta))}{\tau_0^{\alpha}\lambda_1^{\beta-\alpha}\rho^{\frac{1}{\alpha}}(\zeta)}\right)\Delta\zeta\le\omega(t_1)+\frac{p_0^{\beta}}{\tau_0}\nu(t_1), \tag{22}$$

which contradicts (2).
Case(ii): $\alpha>\beta$. Define the function ω by (11). Then (14) holds. Since $z^{\Delta}>0$, there exists a constant $\lambda_2>0$ such that $r(t)(z^{\Delta}(t))^{\alpha}\le r(\tau(t))(z^{\Delta}(\tau(t)))^{\alpha}\le\lambda_2$, hence

$$(z^{\Delta}(t))^{\frac{\beta-\alpha}{\beta}}\ge\left(\frac{r(t)}{\lambda_2}\right)^{\frac{\alpha-\beta}{\alpha\beta}} \tag{23}$$

from (11), and (14), we obtain

$$\omega^{\Delta}(t)\le\rho(t)\frac{[r(t)(z^{\Delta}(t))^{\beta}]^{\Delta}}{z^{\alpha}(\tau(t))}+\frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t))-\frac{\beta\tau_0\rho(t)r^{\frac{\beta-\alpha}{\alpha\beta}}(t)r^{-1/\alpha}(\tau(t))}{\rho^{\frac{\beta+1}{\beta}}(\sigma(t))(z^{\Delta}(t))^{\frac{\alpha-\beta}{\beta}}}\omega^{\frac{\beta+1}{\beta}}(\sigma(t)) \tag{24}$$

This with (23) leads to

$$\omega^{\Delta}(t)\le\rho(t)\frac{[r(t)(z^{\Delta}(t))^{\beta}]^{\Delta}}{z^{\beta}(\tau(t))}+\frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t))-\frac{\beta\tau_0\rho(t)\lambda_2^{\frac{\beta-\alpha}{\alpha\beta}}}{\rho^{\frac{\beta+1}{\beta}}(\sigma(t))r^{1/\alpha}(\tau(t))}\omega^{\frac{\beta+1}{\beta}}(\sigma(t)) \tag{25}$$

Applying Lemma 1.4, we conclude

$$\omega^{\Delta}(t)\le\rho(t)\frac{[r(t)(z^{\Delta}(t))^{\beta}]^{\Delta}}{z^{\beta}(\tau(t))}+\frac{(\rho^{\Delta}(t))^{\frac{\beta+1}{\beta}}\lambda_2^{\frac{\alpha-\beta}{\alpha}}r^{\beta/\alpha}(\tau(t))}{(\beta+1)^{\beta+1}\tau_0^{\beta}\rho^{\beta}(t)} \tag{26}$$

On the other hand, define ν as in (17). Similarly, we have

$$\nu^{\Delta}(t)\le\rho(t)\frac{[r(t)(z^{\Delta}(t))^{\beta}]^{\Delta}}{z^{\beta}(\tau(t))}+\frac{(\rho^{\Delta}(t))^{\frac{\beta+1}{\beta}}\lambda_2^{\frac{\alpha-\beta}{\alpha}}r^{\beta/\alpha}(\tau(t))}{(\beta+1)^{\beta+1}\tau_0^{\beta}\rho^{\beta}(t)} \tag{27}$$

By virtue of (10), (26), and (27), we deduce that

$$\begin{aligned}\omega^{\Delta}(t)+\frac{p_0^{\beta}}{\tau_0}\nu^{\Delta}(t)&\le\frac{\rho(t)}{z^{\beta}(\tau(t))}\left([r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}+\frac{p_0^{\beta}}{\tau_0}[r(\tau(t))(z^{\Delta}(\tau(t)))^{\alpha}]^{\Delta}\right)\\&\quad+\left(1+\frac{p_0^{\beta}}{\tau_0}\right)\frac{(\rho^{\Delta}(t))^{\frac{\beta+1}{\beta}}\lambda_2^{\frac{\alpha-\beta}{\alpha}}r^{\beta/\alpha}(\tau(t))}{(\beta+1)^{\beta+1}\tau_0^{\beta}\rho^{\beta}(t)}\\&\le-K_0\rho(t)Q(t)\frac{z^{\beta}(\delta(t))}{z^{\beta}(\tau(t))}+\left(1+\frac{p_0^{\beta}}{\tau_0}\right)\frac{(\rho^{\Delta}(t))^{\frac{\beta+1}{\beta}}\lambda_2^{\frac{\alpha-\beta}{\alpha}}r^{\beta/\alpha}(\tau(t))}{(\beta+1)^{\beta+1}\tau_0^{\beta}\rho^{\beta}(t)}\end{aligned} \tag{28}$$

Since $\delta(t)\ge\tau(t)$ and $z^{\Delta}(t)>0$, then we obtain

$$\omega^{\Delta}(t)+\frac{p_0^{\beta}}{\tau_0}\nu^{\Delta}(t)\le-K_0\rho(t)Q(t)+\left(1+\frac{p_0^{\beta}}{\tau_0}\right)\frac{(\rho^{\Delta}(t))^{\frac{\beta+1}{\beta}}\lambda_2^{\frac{\alpha-\beta}{\alpha}}r^{\beta/\alpha}(\tau(t))}{(\beta+1)^{\beta+1}\tau_0^{\beta}\rho^{\beta}(t)} \tag{29}$$

Consequently,

$$\int_{t_1}^{t}\left(K_0\rho(\zeta)Q(\zeta)-\left(1+\frac{p_0^{\beta}}{\tau_0}\right)\frac{(\rho^{\Delta}(\zeta))^{\frac{\beta+1}{\beta}}\lambda_2^{\frac{\alpha-\beta}{\alpha}}r^{\beta/\alpha}(\tau(\zeta))}{(\beta+1)^{\beta+1}\tau_0^{\beta}\rho^{\beta}(\zeta)}\right)\Delta\zeta\le\omega(t_1)+\frac{p_0^{\beta}}{\tau_0}\nu(t_1) \tag{30}$$

this contradicts (3). The proof is complete. □

Example 2.2. *Consider the second-order neutral differential equation*

$$\big(x(t)+\frac{1}{2}x(t-1)\big)''+\frac{\gamma}{t^2}x(t)=0 \quad for \quad t\ge 1. \tag{31}$$

where $\gamma>0$ *is a constant,* $r(t)=1$, $p_0=\frac{1}{2}$, $\tau(t)=t-1$, $\tau_0=1$, $Q(t)=\frac{\gamma}{t^2}$ *and* $\delta(t)=t$*.Clearly,* $\alpha=\beta=1$ *and* $\tau(t)\ge\delta(t)$*. Choose* $\rho(t)=t$*, then by* (2)

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\left(K_0\rho(\zeta)Q(\zeta)-\left(1+\frac{p_0^{\beta}}{\tau_0}\right)\frac{(\alpha/\beta)^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(\rho^{\Delta}(\zeta))^{\alpha+1}r(\tau(\zeta))}{\tau_0^{\alpha}\lambda_1^{\beta-\alpha}\rho^{\frac{1}{\alpha}}(\zeta)}\right)\Delta\zeta\\&\qquad=\limsup_{t\to\infty}\int_{t_0}^{t}\left(K_0\frac{\gamma}{\zeta}-\left(\frac{3}{8}\right)\frac{1}{\zeta}\right)d\zeta=\infty\end{aligned} \tag{32}$$

provided that $\gamma>\frac{3}{8K_0}$*. Hence,* (31) *is oscillatory if* $\gamma>\frac{3}{8K_0}$ *. For any* $K_0>1$ *our result is better than results obtained in [13].*

Example 2.3. *Consider the second order neutral differential equation*

$$\left[t^{1/2}(x(t)+p_0x(\gamma t))'\right]'+\frac{a}{t^{3/2}}x(\mu t)=0 \tag{33}$$

where $0<\gamma<\infty$, $0<\mu<1$ *and* $a>0$*. Here* $r(t)=t^{1/2}$, $0<p_0<\infty$, $\tau(t)=\gamma t$, $\tau_0=\gamma$, $q(t)=Q(t)=\frac{a}{t^{3/2}}$ *and* $\delta(t)=\mu t$*. Here* $\alpha=\beta=1$ *. In [3], the authors studied this example in some cases for* γ,β*, but they didn't get results in case of* $\tau(t)<\delta(t)<t$ *, to obtain this case choose* $0<\gamma<\mu<1$ *and* $\rho(t)=t$ *. Application of* (2)*, then we get*

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\left(K_0\rho(\zeta)Q(\zeta)-\left(1+\frac{p_0^{\beta}}{\tau_0}\right)\frac{(\alpha/\beta)^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(\rho^{\Delta}(\zeta))^{\alpha+1}r(\tau(\zeta))}{\tau_0^{\alpha}\lambda_1^{\beta-\alpha}\rho^{\frac{1}{\alpha}}(\zeta)}\right)\Delta\zeta\\&\qquad=\limsup_{t\to\infty}\int_{t_0}^{t}\left(\frac{a}{\zeta^{3/2}}\zeta-\frac{1}{4}\left(1+\frac{p_0}{\gamma}\right)\frac{1}{\sqrt{\gamma}\sqrt{\zeta}}\right)d\zeta=\infty\end{aligned} \tag{34}$$

Hence, (33) *is oscillatory if* $a>\frac{1}{4}\left(1+\frac{p_0}{\gamma}\right)\frac{1}{\sqrt{\gamma}}$*.*

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